

Decomposition of Arbitrarily Shaped Morphological Structuring Elements

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Abstract —For image processing systems that have a limited size of region of support, say 3×3 , direct implementation of morphological operations by a structuring element larger than the prefixed size is impossible. The decomposition of morphological operations by a large structuring element into a sequence of recursive operations, each using a smaller structuring element, enables the implementation of large morphological operations. In this paper, we present the decomposition of arbitrarily shaped (convex or concave) structuring elements into 3×3 elements, optimized with respect to the number of 3×3 elements. The decomposition is based on the concept of factorization of a structuring element into its prime factors. For a given structuring element, all its corresponding 3×3 prime concave factors are first determined. From the set of the prime factors, the decomposability of the structuring element is then established, and subsequently the structuring element is decomposed into a smallest possible set of 3×3 elements. Examples of optimal decomposition and structuring elements that are not decomposable are presented.

Index Terms — Mathematical morphology, structuring element decomposition, concave boundary.

I. INTRODUCTION

THE chain rule of dilation enables a single dilation to be decomposed into a sequence of recursive dilation,¹ which is viewed as the decomposition of a large structuring element into a set of smaller elements [1]. The decomposition of structuring elements plays an important role in the implementation of morphological operations. For image processing systems that have a limited size of region of support, direct implementation of dilation by a structuring element larger than the prefixed size is impossible. Structuring element decomposition enables the dilation by a large structuring element to be implemented equivalently as recursive dilations by a number of smaller elements. In addition, structuring element decomposition often speeds up the processing on the systems which have no practical limit on the size of region of support.

A number of researchers have noted the importance of the decomposition of structuring elements [1-3], and there have been studies in the decomposition of structuring elements,

¹ The same is true for erosion.

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each proposing a different algorithm. Zhuang and Haralick [4] developed an optimal algorithm for the decomposition of an arbitrary structuring element into two-pixel elements, each with an arbitrary size of region of support. Xu [5] developed an optimal algorithm for the decomposition of *convex* structuring elements for systems with a 3×3 region of support such as the Cytocomputer [6]. Park and Chin [7] proposed an optimal algorithm for the decomposition of *convex* structuring elements for 4-connected parallel array processors using the number of shifts as the optimization criterion. Several others [8-10] have investigated the decomposition of structuring elements, but are limited to convex or other restrictive shapes.

In this paper, an optimal decomposition of simply connected² binary structuring elements of *arbitrary* shape into 3×3 elements is proposed. For a given simply connected binary structuring element S , the decomposition of S is given by

$$S = A^1 \oplus A^2 \oplus \dots \oplus A^n,$$

where A^i is 3×3 or less and simply connected. Such a decomposition makes dilation of an image X by S possible using a 3×3 region of support in a recursive manner, given by

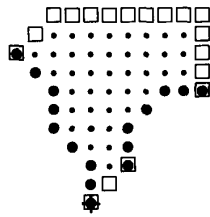
$$X \oplus S = (\dots (X \oplus A^1) \oplus A^2) \oplus \dots \oplus A^n).$$

However, not all structuring elements can be decomposed into 3×3 elements. Hence, it is required to first determine the decomposability of S . If S is decomposable, then an optimal decomposition of S is sought for.

In Section II, terminologies and notations are defined. In Section III, a number of necessary conditions for decomposition are derived to narrow down the set of all images to a smaller set which contains all the decomposable images. In Section IV, the concept of factorization is introduced which is analogous to the factorization of integers. All possible 3×3 prime concave factors are then determined and represented by chain code. Finally, in Section V, a procedure based on factorization is defined to first determine the decomposability and subsequently the optimal decomposition. Some examples of the optimal decomposition are provided in Section VI. All proofs of propositions are presented in Appendix A.

² A binary image is simply connected if it is 8-connected and contains no holes.

³ In this paper, the boundary refers to the exterior boundary, i.e., the boundary of hole, if it exists, is not considered.



- : a pixel in concave boundary
- : a pixel in convex boundary
- : a pixel in both concave and convex boundaries
- ⊕ : a starting pixel of chain code

Chain code : 11(221100)22224444444455(7766)(7766)

Fig. 1. An example of concave and convex boundaries.

II. DEFINITIONS AND NOTATIONS

Definition 1: A corner of a connected binary image is a boundary pixel³ which is 8-connected to its two adjacent boundary pixels by directions i and j in Freeman's chain code [11] where $i \neq j$. Note that a corner is defined by three boundary pixels with two distinct directions in chain code. □

Definition 2: The angle of a corner is the difference between the two directions of the three-pixel corner, measured from inside the image. A concave corner is a corner whose angle is larger than 180°. A convex corner is a corner whose angle is less than 180°. □

Definition 3: A boundary segment is the set of connected boundary pixels from a convex corner to the next convex corner, including the two convex corners. A concave boundary is a boundary segment which contains one or more concave corners. A convex boundary is a boundary segment which contains no concave corners. □

Note that each concave or convex boundary has one boundary pixel common to a neighboring concave or convex boundary. Therefore, the chain code of a boundary of connected binary image is divided into non-overlapping concave and convex boundaries. Without loss of generality, we assume that the chain code starts at a convex corner with 0 direction and runs counterclockwise. In addition, parentheses are used to distinguish concave boundaries from convex boundaries. See Fig. 1 for an example of concave and convex boundaries and the chain code representation of binary image.

A. Concave Boundaries. There exist an infinite number of different concave boundaries, but if the images are restricted to the size of 3×3 , there are only 28 distinct concave boundaries. Each of the 28 concave boundaries is denoted by Q_{Ti} as defined in Fig. 2. The subscript T denotes the type of concave boundary, and i denotes the starting chain code direction of

boundary. There are five types, i.e., $T = U, J, L, V,$ or r . For example, Q_{U2} denotes concave boundary of Type U , which starts with direction 2, and $Q_{U2} = (2176)$. The set $\{Q_{Ti}\}$ denotes the entire collection of all 28 concave boundaries.

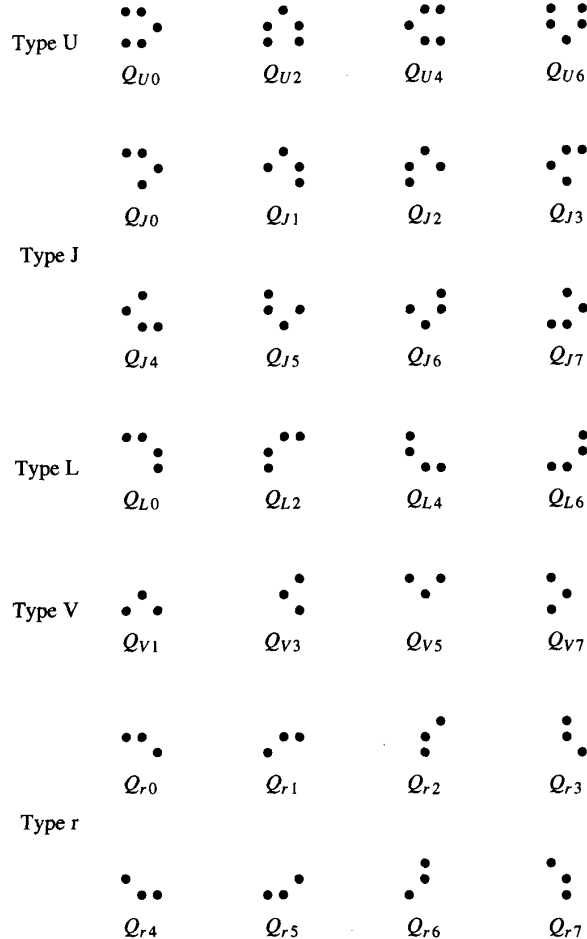


Fig. 2. There are only 28 concave boundaries for 3×3 images. Each Q_{Ti} denotes a concave boundary. The first subscript denotes the boundary type; the second denotes the starting chain code direction.

Definition 4: An image X is equivalent to an image Y , denoted by $X \sim Y$ if and only if X and Y are identical except for a translation. □

Definition 5: An image A is a factor of an image S if and only if $S = A \oplus B$ for some image B . A factor A of S is a trivial factor if and only if $A \sim S$ or A is a one-pixel image. A factor A of S is a prime factor if and only if every factor of A is a trivial factor. □

Definition 5 for binary images is analogous to that of integers. For example, 1, 2, 3, 4, 6, and 12 are factors of 12; 1 and 12 are trivial factors; and 2 and 3 are prime factors.

Definition 6: A simply connected binary image S is decomposable if and only if S can be represented by

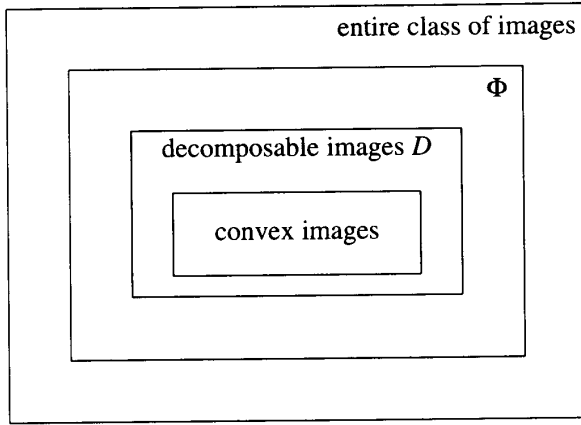
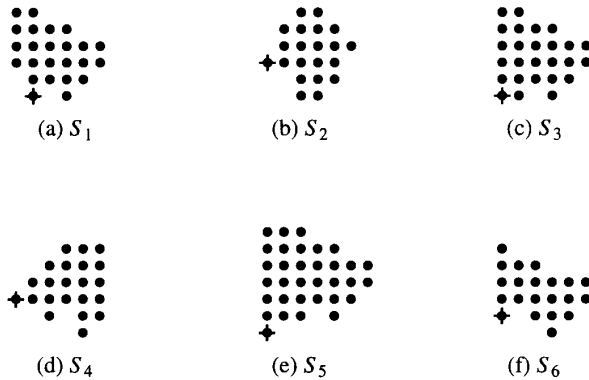


Fig. 3. Relationship between F and D.

$$S = A^1 \oplus A^2 \oplus \dots \oplus A^n, \quad (1)$$

where A^i is 3×3 or less and simply connected. (1) is called a *decomposition* of S , and is an *optimal decomposition* of S when n is minimized. \square

Fig. 4. Examples of members and non-members of Φ .

(a) $S_1 = Q_{V1}1^22Q_r^246^3Q_{r1} \in \Phi$. (b) $S_2 = Q_{L0}01Q_{r2}3^245Q_{r0} \in \Phi$. (c) $S_3 = 0Q_{V1}1^22Q_r^246^3 \notin \Phi$ because $s_{V1}S_0 \neq 0$ which violates (6). (d) $S_4 = Q_{r0}Q_{r1}12^44^25^3 \notin \Phi$ because $s_{r0}S_{r1} \neq 0$ which violates (4). (e) $S_5 = Q_{r1}Q_{V1}1^22Q_r^24^26^6 \notin \Phi$ because the order of Q_{r1} and Q_{V1} does not satisfy (2). (f) $S_6 = (177)1^22(443)Q_{r4}6^4 \notin \Phi$ because (177) and (443) are not members of $\{Q_{r1}\}$. (+ indicates the starting pixel of chain code for each image.)

Note that every A^i in (1) is a factor of S , and the decomposition of S can be thought of as the determination of its factors for dilations, which is analogous to the factorization of integers.

B. Decomposability. The concept of factorization is used in the decomposition; however, not every arbitrarily shaped structuring element S is decomposable. Therefore, it is essential to first determine the decomposability of S , that is, to verify $S \in D$ where D is a set containing only all decomposable images.

There are images that are obviously not decomposable. The elimination of those images from the set of *all* images yields a

smaller set Φ . Set D and set Φ are not identical. Set Φ contains both "decomposable" and "non-decomposable" images, while D contains only all decomposable images; that is, $D \subset \Phi$ as depicted in Fig. 3, where convex images are defined as images which contain no concave boundaries. For example, S_2 in Fig. 4 is a member of Φ , but not a member of D because it is not decomposable.

In reality, we are not seeking for the full Φ set; instead, for each S in question, we first verify $S \in \Phi$ and subsequently $S \in D$. If the verification fails, S is not decomposable; therefore recursive implementation of morphological operations using S is not possible. In the following, we first define the set Φ in Definition 7 and show that under this definition of Φ , $D \subset \Phi$ in Proposition 4 in the next section.

Definition 7: A connected binary image S is said to be a member of Φ if and only if (i) chain code representation of S has the form of

$$S = Q_{U0}^{S_{U0}} Q_{J0}^{S_{J0}} Q_{L0}^{S_{L0}} Q_{r0}^{S_{r0}} 0^{S_0} Q_{J1}^{S_{J1}} Q_{V1}^{S_{V1}} Q_{r1}^{S_{r1}} 1^{S_1} \dots Q_{J7}^{S_{J7}} Q_{V7}^{S_{V7}} Q_{r7}^{S_{r7}} 7^{S_7}, \quad (2)$$

where the superscripts s_i and s_{r_i} denote the repetition of direction i in convex boundary and concave boundary Q_{r_i} , respectively,⁴ and (ii) the chain code superscripts satisfy the following:

$$s_{U_i}(s_{i-1} + s_{i-2} + s_{i-3} + s_{r(i-1)} + s_{r(i-2)} + s_{V(i-1)}) = 0, \quad (3)$$

$$s_{J_i}(s_{i-1} + s_{i-2} + s_{r(i-1)}) = 0, \quad (4)$$

$$s_{L_i} s_{i-1} = 0, \quad (5)$$

$$s_{V_i} s_{i-1} = 0. \quad (6)$$

\square

Equation (2) implies that the boundary of $S \in \Phi$ consists of both Q_{r_i} and convex boundaries which are arranged in a specific order. Equations (3), (4), (5), and (6) constrain the boundary types which can be contained in $S \in \Phi$ simultaneously. For example, if $S \in \Phi$ contains Q_{r1} , i.e., $s_{r1} \neq 0$, then from (4), $s_0 = s_7 = s_{r0} = 0$. Therefore, S cannot contain directions 0, 7, and concave boundary Q_{r0} . Note that all 3×3 connected images are members of Φ . See Fig. 4 for examples of members and non-members of Φ .

A boundary of connected image in Φ is uniquely defined by its chain code superscripts, where the form of chain code representation is specified in (2). Therefore, the chain code superscripts are used to represent the boundary of image in Φ throughout the rest of the paper.

III. NECESSARY CONDITION FOR DECOMPOSITION

Again, let D be a set which contains only all the decomposable images. In this section, we shall show that every decomposable image is a member of Φ , i.e., $D \subset \Phi$.

We consider $S = A \oplus B$, where $A, B \in \Phi$, and B is 3×3 . Proposition 1 shows that chain code representation of S has the form defined in (2). Propositions 2 and 3 show that the chain

⁴ A zero superscript means an absence of corresponding convex or concave boundary.

code superscripts of S satisfy Equations (3)-(6). From these propositions, we show in Proposition 4 that every decomposable image is a member of Φ . The following gives the details.

Proposition 1: Let $A, B \in \Phi$, and B be 3×3 . If $S = A \oplus B$, then chain code representation of S has the form defined in (2). \square

In Proposition 1, it is shown that dilation of an image in Φ by a 3×3 image does not create concave boundaries other than those given in (2). Furthermore, it shows that the order of convex and concave boundaries of S conforms to that in (2). However, Equations (3)-(6) must also be satisfied to prove that S is a member of Φ , which is given by the following propositions.

Proposition 2: Let $A, B \in \Phi$ and B be 3×3 . Let $S = A \oplus B$. (1) If S contains Q_{U_i} , then

$$a_{U_i} + b_{U_i} \neq 0, \quad (7)$$

$$(a_{i-1} + a_{i-2} + a_{i-3} + a_{r(i-1)} + a_{r(i-2)} + a_{v(i-1)}) + (b_{i-1} + b_{i-2} + b_{i-3} + b_{r(i-1)} + b_{r(i-2)} + b_{v(i-1)}) = 0. \quad (8)$$

(2) If S contains Q_{J_i} , then

$$a_{J_i} + b_{J_i} \neq 0, \quad (9)$$

$$(a_{i-1} + a_{i-2} + a_{r(i-1)}) + (b_{i-1} + b_{i-2} + b_{r(i-1)}) = 0. \quad (10)$$

(3) If S contains Q_{L_i} , then

$$a_{L_i} + b_{L_i} \neq 0, \quad (11)$$

$$a_{i-1} + b_{i-1} = 0. \quad (12)$$

(4) If S contains Q_{V_i} , then

$$(a_i + a_{i-2} + a_{r_i} + a_{r(i-1)} + a_{v_i}) + (b_i + b_{i-2} + b_{r_i} + b_{r(i-1)} + b_{v_i}) \neq 0, \quad (13)$$

$$a_{i-1} + b_{i-1} = 0. \quad (14)$$

(5) If S contains Q_{r_i} , then

$$(a_i + a_{i-1} + a_{r_i}) + (b_i + b_{i-1} + b_{r_i}) \neq 0. \quad (15)$$

(6) If S contains convex boundary of direction i , then

$$a_i + b_i \neq 0. \quad \square \quad (16)$$

Proposition 3: Let $A, B \in \Phi$, and B be 3×3 . If $S = A \oplus B$, then Equations (3)-(6) of S are satisfied. \square

Proposition 3 together with Proposition 1 prove that $S = A \oplus B \in \Phi$, if $A, B \in \Phi$, and B is 3×3 because dilation between connected images produces a connected image. In other words, Φ is closed under dilation by 3×3 image. Now, we are ready to show that every decomposable image is a member of Φ .

Proposition 4: If S is decomposable, i.e., $S = A^1 \oplus A^2 \oplus \dots \oplus A^n$, where A^i is simply connected and 3×3 or less, then $S \in \Phi$. \square

Using Proposition 4, S is said to be non-decomposable if S is not a member of Φ . Hence, non-decomposable images can be identified using their chain code representations. For example, $S_3 - S_6$ in Fig. 4 are not decomposable. However, it should be

noted that *not* all elements of Φ are decomposable since Proposition 4 only gives the necessary condition, but not the sufficient condition.

Since non-decomposable images are of no interest in this paper, only members of Φ will be considered hereafter. We need, however, additional conditions on S to identify D in Φ because Φ also includes non-decomposable images. The determination of prime factors in Φ is used for this purpose and is discussed in the next section.

IV. DETERMINATION OF PRIME CONCAVE FACTORS

This section describes the determination of the 3×3 prime concave factors in Φ . Using these prime factors, the reduction of Φ to D is possible, which is described in Section V.

Because dilation is a union of shifted images, concave boundaries may disappear after dilation. First, the relationship between dilation and chain code is determined in Proposition 5 when none of concave boundaries are removed after dilation. In fact, the removal of a particular Q_{T_i} in A due to dilation by B depends on B 's boundary types. The dependency on B of each boundary type in A is then determined in Proposition 6, from which properties of chain code required to determine factors are derived (Proposition 7). Finally, conditions for the determination of factors are defined in Proposition 8, and all 3×3 prime concave images are listed in Table I. The following gives the details:

Proposition 5: Let $A, B, S \in \Phi$, and B be 3×3 . If $S = A \oplus B$ and all concave boundaries in A and B are contained in S (i.e., none of concave boundaries in A and B are removed), then $s_i = a_i + b_i$ and $s_{T_i} = a_{T_i} + b_{T_i}$. \square

The converse of Proposition 5 is not true in general because chain code specifies only boundary. However, if we consider only boundaries of $A \oplus B$ and S , the converse is true, which is stated in Proposition 7. To derive Proposition 7, Proposition 6 is necessary.

Proposition 6: Let $A, B, S \in \Phi$ and $S = A \oplus B$. If $a_{T_i} \neq 0$ but $s_{T_i} = 0$, that is, A contains Q_{T_i} but S does not, then the following conditions for B must be true for each type of concave boundary Q_{T_i} in A which is removed:

$$b_{i-1} + b_{i-2} + b_{i-3} + b_{r(i-1)} + b_{r(i-2)} + b_{v(i-1)} \neq 0 \quad \text{for Type } U, \quad (17)$$

$$b_{i-1} + b_{i-2} + b_{r(i-1)} \neq 0 \quad \text{for Type } J, \quad (18)$$

$$b_{i-1} \neq 0 \quad \text{for Type } L \text{ and } V \quad (19)$$

\square

Proposition 7: Let $A, B, S \in \Phi$ and B be 3×3 . If $s_i = a_i + b_i$ and $s_{T_i} = a_{T_i} + b_{T_i}$, then the boundary of S is identical to the boundary of $A \oplus B$. \square

For the simple case when S and B are convex and S is simply connected, it has been shown that if $s_i \geq b_i$ and $A \oplus B$ is simply connected where A is a convex image defined by chain code superscript $a_i = s_i - b_i$, then $S = A \oplus B$; B is referred to as a factor of S [7]. Now, if S and B are concave, similar chain code arithmetic involving $S = A \oplus B$ is defined using Proposi-

TABLE I.

ALL 3×3 PRIME CONCAVE IMAGES CONTAINING CONCAVE BOUNDARIES Q_{U0} , Q_{L0} , Q_{J1} , Q_{L0} , Q_{V1} , Q_{R0} , AND Q_{R1} ARE GIVEN IN CHAIN CODES. CONCAVE IMAGES CONTAINING Q_{U1} , Q_{L1} , Q_{J2} , AND Q_{R2} FOR $i = 2, 4, 6$ ARE OBTAINED BY ROTATING Q_{U0} , Q_{L0} , Q_{J1} , AND Q_{R0} BY $i \times 45^\circ$, I.E., BY INCREASING EACH CHAIN CODE BY i UNITS, RESPECTIVELY. CONCAVE IMAGES CONTAINING Q_{U3} , Q_{V3} , AND Q_{R3} FOR $i = 3, 5, 7$ ARE OBTAINED BY ROTATING Q_{J1} , Q_{V1} , AND Q_{R1} BY $(i - 1) \times 45^\circ$, I.E., BY INCREASING EACH CHAIN CODE BY $(i - 1)$ UNITS, RESPECTIVELY.

Q_{U0}	Q_{L0}	$Q_{V1}Q_{V3}4^26^2$	Q_{R0}	Q_{R1}
$Q_{U0}0^22^24^2$	$Q_{L0}2^24^2$	$Q_{V1}Q_{V3}4^2Q_{V7}$	$Q_{R0}2^245$	$Q_{R1}24^2Q_{V7}$
$Q_{U0}0^2234$	$Q_{L0}234$	$Q_{V1}Q_{V3}456$	$Q_{R0}2^2Q_{V5}6$	$Q_{R1}24Q_{R6}$
$Q_{U0}0124^2$	Q_{V1}	$Q_{V1}Q_{V3}4Q_{R6}$	$Q_{R0}2^2Q_{R5}$	$Q_{R1}2Q_{V5}6^2$
$Q_{U0}0134$	$Q_{V1}2^24^2Q_{V7}$	$Q_{V1}Q_{V3}Q_{V5}6^2$	$Q_{R0}235$	$Q_{R1}2Q_{V5}Q_{V7}$
Q_{J0}	$Q_{V1}2^24Q_{R6}$	$Q_{V1}Q_{V3}Q_{V5}Q_{V7}$	$Q_{R0}2Q_{R4}6$	$Q_{R1}2Q_{R5}6$
$Q_{J0}0^22^24^2$	$Q_{V1}2^2Q_{V5}6^2$	$Q_{V1}Q_{V3}Q_{R5}6$	$Q_{R0}24^2$	$Q_{R1}25^2$
$Q_{J0}0^2234$	$Q_{V1}2^2Q_{V5}Q_{V7}$	$Q_{V1}Q_{V3}5^2$	$Q_{R0}Q_{V3}4^26$	$Q_{R1}346^2$
$Q_{J0}124^2$	$Q_{V1}2^2Q_{R5}6$	$Q_{V1}Q_{R3}46^2$	$Q_{R0}Q_{V3}45$	$Q_{R1}34Q_{V7}$
$Q_{J0}134$	$Q_{V1}2^25^2$	$Q_{V1}Q_{R3}4Q_{V7}$	$Q_{R0}Q_{V3}Q_{V5}6$	$Q_{R1}356$
Q_{J1}	$Q_{V1}234Q_{V7}$	$Q_{V1}Q_{R3}56$	$Q_{R0}Q_{V3}Q_{R5}$	$Q_{R1}3Q_{R6}$
$Q_{J1}2^24^26$	$Q_{V1}23Q_{R6}$	$Q_{V1}Q_{R3}Q_{R6}$	$Q_{R0}Q_{R3}5$	$Q_{R1}Q_{R4}6^2$
$Q_{J1}2^245$	$Q_{V1}2Q_{R4}6^2$	$Q_{V1}3^26^2$	$Q_{R0}Q_{R3}46$	$Q_{R1}Q_{R4}Q_{V7}$
$Q_{J1}2346$	$Q_{V1}2Q_{R4}Q_{V7}$	$Q_{V1}3^2Q_{V7}$	$Q_{R0}3^26$	$Q_{R1}4^26$
$Q_{J1}235$	$Q_{V1}24^26$	$Q_{V1}346$	$Q_{R0}34$	$Q_{R1}45$
	$Q_{V1}245$	$Q_{V1}35$		

tion 7 and Proposition 8.

Proposition 8: Let $S, B \in \Phi$, B be 3×3 , and S be simply connected. If (i) superscripts $a_i = s_i - b_i$ and $a_{T_i} = s_{T_i} - b_{T_i}$ define valid chain code for an image, and (ii) $A \oplus B$ is simply connected where A is defined by a_i and a_{T_i} , then $S = A \oplus B$ and B is a factor of S . \square

Note that Proposition 8 only gives sufficient condition for factors, and there are many factors which do not satisfy Proposition 8. However, factors which do not satisfy Proposition 8 are not essential to the decomposition. To justify this argument, we need to use Proposition 9.

Proposition 9: Let $S, A, B \in \Phi$, B be 3×3 , and S be simply connected. Let $S = A \oplus B$. If $b_{T_i} \neq 0$ and $s_{T_i} = 0$, that is, B contains Q_{T_i} but S does not, then there exists a \hat{B} such that $S = A \oplus \hat{B}$ and all concave boundaries in \hat{B} are contained in S . \square

Proposition 9 implies that if S is decomposable, then S can be represented by $S = A^1 \oplus \dots \oplus A^n$, where all concave boundaries in A^i , $i = 1, 2, \dots, n$ are contained in S , because whenever

concave boundary in a particular 3×3 factor is removed, this factor can be replaced by a different one such that all concave boundaries in the new one are contained in S . Fig. 5 shows an example of Proposition 9: Q_{V5} in A^1 and Q_{R3} in A^2 are not contained in S . Then, \hat{A}^1 and \hat{A}^2 replace A^1 and A^2 , respectively, such that all concave boundaries in \hat{A}^1 and \hat{A}^2 are contained in S .

Suppose that $S = A^1 \oplus \dots \oplus A^n$ and A^1 does not satisfy Proposition 8. After replacing each A^i by \hat{A}^i such that all concave boundaries in \hat{A}^i are contained in S , we have $S = \hat{A}^1 \oplus \dots \oplus \hat{A}^n$. Let $\hat{A}^2 \oplus \dots \oplus \hat{A}^n = C \in \Phi$. Then, from Proposition 5, $s_i = c_i + \hat{a}_i$ and $s_{T_i} = c_{T_i} + \hat{a}_{T_i}$. Therefore, \hat{A}^1 satisfies Proposition 8. In general, any decomposable S can be decomposed such that all factors satisfy Proposition 8. Moreover, \hat{A}^1 and A^1 have identical contribution to the decomposition except for the number of pixels involved. But, the criterion of our decomposition is the number of 3×3 elements and not the total number of pixels. Thus, \hat{A}^1 and A^1 can be thought of as an identical factor for our criterion. Therefore, those factors which do not satisfy Proposition 8 need not be considered, and only factors which satisfy Proposition 8 are essential to the de-

$$S = \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \\ \bullet \bullet \bullet \bullet \bullet \\ \bullet \bullet \bullet \bullet \bullet \\ \bullet \bullet \bullet \bullet \bullet \\ \bullet \bullet \bullet \bullet \bullet \end{array} = \begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \end{array} \oplus \begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \end{array} \oplus \begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \end{array} = A^1 \oplus A^2 \oplus A^3$$

$$= \begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \end{array} \oplus \begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \end{array} \oplus \begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \end{array} = \hat{A}^1 \oplus \hat{A}^2 \oplus A^3$$

Fig. 5. Proposition 2: Replace A^1 by \hat{A}^1 and A^2 by \hat{A}^2 . Before replacement, boundaries Q_{15} in A^1 and Q_{16} in A^2 are not contained in S . After re-placement, boundaries Q_{11} and Q_{12} are contained in S . (+ indicates the origin.)

composition.

Now, we propose a procedure to determine all necessary 3×3 prime concave factors of simply connected $S \in \Phi$. There are a limited number of 3×3 prime images which contain a particular concave boundary type — Type U has 4 distinct 3×3 prime images; Type J has 4; Type L has 2; Type V has 28; and Type r has 14. These images are given in Table I in chain codes. For each concave boundary of a given S , Proposition 8 and Table I are used to determine the 3×3 prime concave factors of S . For example, if S has concave boundary Q_{11} , then all 14 prime images in the last column of Table I are examined to identify the 3×3 prime concave factors which contain Q_{11} . Finally, we have another necessary condition for decomposition given by Proposition 10.

Proposition 10: If $S \in \Phi$ is decomposable, then for each concave boundary Q_{Ti} in S , there exists at least one 3×3 concave factor of S which contains the same Q_{Ti} . \square

Note that even when every concave boundary in S has its corresponding 3×3 prime concave factors, S is not always decomposable. See Fig. 6 for an example. This implies that the converse of Proposition 10 is not true; it gives only a necessary condition for decomposition.

The determination of all 3×3 prime concave factors of S provides a simple procedure to determine the set D and subsequently the decomposition of $S \in D$, which is described in the next section.

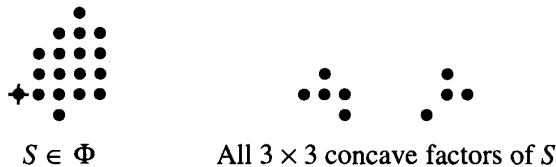


Fig. 6. $S = Q_{10}Q_{11}2^335^2Q_{16}$. Each concave boundary in S has a corresponding 3×3 concave factor, but S is not decomposable. (+ indicates the starting pixel of chain code.)

V. DECOMPOSITION INTO 3×3 ELEMENTS

In this section, a procedure to determine the decomposability of an image in Φ is developed. Such a decomposable image is a member of D . This procedure involves an existence test of the solution of a set of constrained linear equations. In addition, the decomposition of $S \in D$ into 3×3 elements is determined by solving the same linear equations.

In the previous section, we found all necessary 3×3 prime concave factors of a simply connected $S \in \Phi$. Let those factors be the set $\{A^i\}$. If S is decomposable, i.e., $S \in D$, then S is given by

$$S = C^1 \oplus C^2 \oplus \dots \oplus C^m \oplus B,$$

where $C^j \in \{A^i\}$, and B is a convex factor of S which is guaranteed to be decomposable [5]. It remains to check whether or not such selections of C^j 's and B exist (if they exist, then $S \in D$), and to determine these factors if $S \in D$. Now, we formulate this problem into two constrained linear equations, one for concave boundaries and one for convex boundaries.

Suppose $S \in \Phi$ consists of (i) m distinct concave boundaries, V_1, V_2, \dots, V_m , that is, V_k is one of Q_{Ti} where $s_{Ti} \neq 0$, (ii) l distinct chain code directions in the convex boundaries, d_1, d_2, \dots, d_l , that is, d_k is one of the chain code directions i where $s_i \neq 0$, and (iii) n distinct 3×3 prime concave factors, A^1, A^2, \dots, A^n , from Proposition 8 and Table I. For example, for $S = Q_{11}1^22Q_{14}46^3Q_{17}$, V_k and d_k are constructed as follows: $V_1 = Q_{11}$, $V_2 = Q_{14}$, $V_3 = Q_{17}$, $d_1 = 1$, $d_2 = 2$, $d_3 = 4$, and $d_4 = 6$. We define two matrices Θ and Ω , where Θ is $m \times n$ and Ω is $l \times n$, as follows:

$$[\Theta]_{ij} = \text{number of } V_i \text{'s in } A^j$$

$$[\Omega]_{ij} = \text{number of } d_i \text{'s in } A^j.$$

To determine Θ and Ω , each factor A^j is examined and the counts of V_i 's and d_i 's in A^j are determined as entries of the matrices. Furthermore, we define two vectors, \mathbf{Y} of size m and \mathbf{Z} of size l , as follows:

$$[\mathbf{Y}]_i = \text{number of } V_i \text{'s in } S$$

$$[\mathbf{Z}]_i = \text{number of } d_i \text{'s in } S.$$

Vectors \mathbf{Y} and \mathbf{Z} are constructed from the chain code representation of S . Finally, a variable vector is defined given as

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T,$$

where x_i 's are non-negative integers.

Proposition 11: A simply connected $S \in \Phi$ is decomposable if and only if there exists an \mathbf{X} such that

$$\Theta \mathbf{X} = \mathbf{Y}, \quad (20)$$

$$\Omega \mathbf{X} \leq \mathbf{Z}, \quad (21)$$

and $x_1 A^1 \oplus \dots \oplus x_n A^n \oplus B$ is simply connected, where $x_i A^i$ is x_i -fold dilations of A^i and B is a convex factor defined by chain code superscript $b_i = [\mathbf{Z} - \Omega \mathbf{X}]_i$. If such an \mathbf{X} exists, then the decomposition of S is

$$S = x_1 A^1 \oplus x_2 A^2 \oplus \dots \oplus x_n A^n \oplus B. \quad (22)$$

To find an \mathbf{X} satisfying Proposition 11, we first solve Equations (20) and (21), then verify that $x_1 A^1 \oplus \dots \oplus x_n A^n \oplus B$ is simply connected. Note that if at least one of A^i with $x_i \geq 1$ or B is 4-connected, then $x_1 A^1 \oplus \dots \oplus x_n A^n \oplus B$ is always simply connected [7].

To optimize the decomposition in (22) for a particular \mathbf{X} , a procedure adopted from [5] given in Appendix B is used. However, \mathbf{X} satisfying Proposition 11 is not unique, thereby each \mathbf{X} gives a corresponding decomposition of S (Equation (22)) and a corresponding optimized result. To guarantee an optimal decomposition of S , the decomposition of S in (22) for each \mathbf{X} must be optimized by the procedure in Appendix B, and the one with the smallest number of 3×3 elements will be selected for the optimal decomposition.⁵ See Example 2 in Section VI for a detailed illustration.

The search for the optimal decomposition among all possible solutions needs not be exhaustive. There exists a lower bound on the number of 3×3 elements of the decomposition given by $\max\{l_{x_{\max}}(S), l_{x_{\min}}(S), l_{y_{\max}}(S), l_{y_{\min}}(S)\}$, where $x_{\max}(S) = \max\{x \mid (x, y) \in S\}$ and the other three terms are defined accordingly [5]. Therefore, when the decomposition from a particular \mathbf{X} yields this lower bound, an optimal decomposition is guaranteed and the search terminates.

The procedure for an optimal decomposition of arbitrarily shaped S into 3×3 elements is summarized in the following:

Procedure: Optimal Decomposition of S into 3×3 Elements

- 1) Verify $S \in \Phi$. If not, S is not decomposable, i.e., $S \notin D$.
- 2) Determine all 3×3 prime concave factors $\{A^i\}$ of S using Proposition 8 and Table I. If no 3×3 prime concave factor can be determined for any concave boundary in S , then $S \notin D$.
- 3) Define Θ , Ω , \mathbf{Y} , and \mathbf{Z} . Solve $\Theta \mathbf{X} = \mathbf{Y}$ and $\Omega \mathbf{X} \leq \mathbf{Z}$ for \mathbf{X} , a non-negative integer vector.

⁵ The number of solutions is finite because the solutions are restricted to non-negative integers.

- 4) If \mathbf{X} exists and $x_1 A^1 \oplus \dots \oplus x_n A^n \oplus B$ is simply connected, where the convex factor B is given by chain code superscript $b_i = [\mathbf{Z} - \Omega \mathbf{X}]_i$, then $S \in D$ and $S = x_1 A^1 \oplus \dots \oplus x_n A^n \oplus B$. Otherwise, $S \notin D$.
- 5) Search for an optimal decomposition using the optimization procedure in Appendix B.

VI. EXAMPLES

Example 1: A structuring element used most often in the image processing is the circle. Suppose that S is a circle given by $S = Q_{r0}0^2 Q_{r1}Q_{r2}2^2 Q_{r3}Q_{r4}4^2 Q_{r5}Q_{r6}6^2 Q_{r7}$ as in Fig. 7(a).

It has been verified that $S \in \Phi$. Next, we find all the 3×3 prime concave factors for each concave boundary of S using Proposition 8 and Table I. We can see that each concave boundary has at least one factor, and there are 20 such factors in total. A^i is assigned to each of those factors as in Fig. 7(b), and V_i and d_i are constructed as follows:

$$\begin{aligned} V_1 &= Q_{r0}, V_2 = Q_{r1}, V_3 = Q_{r2}, V_4 = Q_{r3}, \\ V_5 &= Q_{r4}, V_6 = Q_{r5}, V_7 = Q_{r6}, V_8 = Q_{r7}, \\ d_1 &= 0, d_2 = 2, d_3 = 4, d_4 = 6. \end{aligned}$$

Then,

$$\Theta = \begin{bmatrix} 111100000000000000 \\ 000011110000000000 \\ 000000001111000000 \\ 0001000000001110000 \\ 01000010000000011000 \\ 10000100000100000100 \\ 00001000010001000010 \\ 00000000100010001001 \end{bmatrix}$$

$$\Omega = \begin{bmatrix} 00000000010112121210 \\ 2110110000000001122 \\ 00211002211010000001 \\ 01010121002100210000 \end{bmatrix}$$

$$\mathbf{Y} = [11111111]^T \text{ and } \mathbf{Z} = [2222]^T.$$

One possible \mathbf{X} satisfying Proposition 11 is

$$\mathbf{X} = [01000100010010000000]^T.$$

Therefore, the decomposition of S exists and is given by

$$S = A^2 \oplus A^6 \oplus A^{10} \oplus A^{13}.$$

After optimization (Appendix B), the decomposition of S given in Fig. 7(c) results. Since this decomposition requires four 3×3 elements and the lower bound = $\max\{l_{x_{\max}}(S), l_{x_{\min}}(S), l_{y_{\max}}(S), l_{y_{\min}}(S)\} = 4$, this is an optimal decomposi-

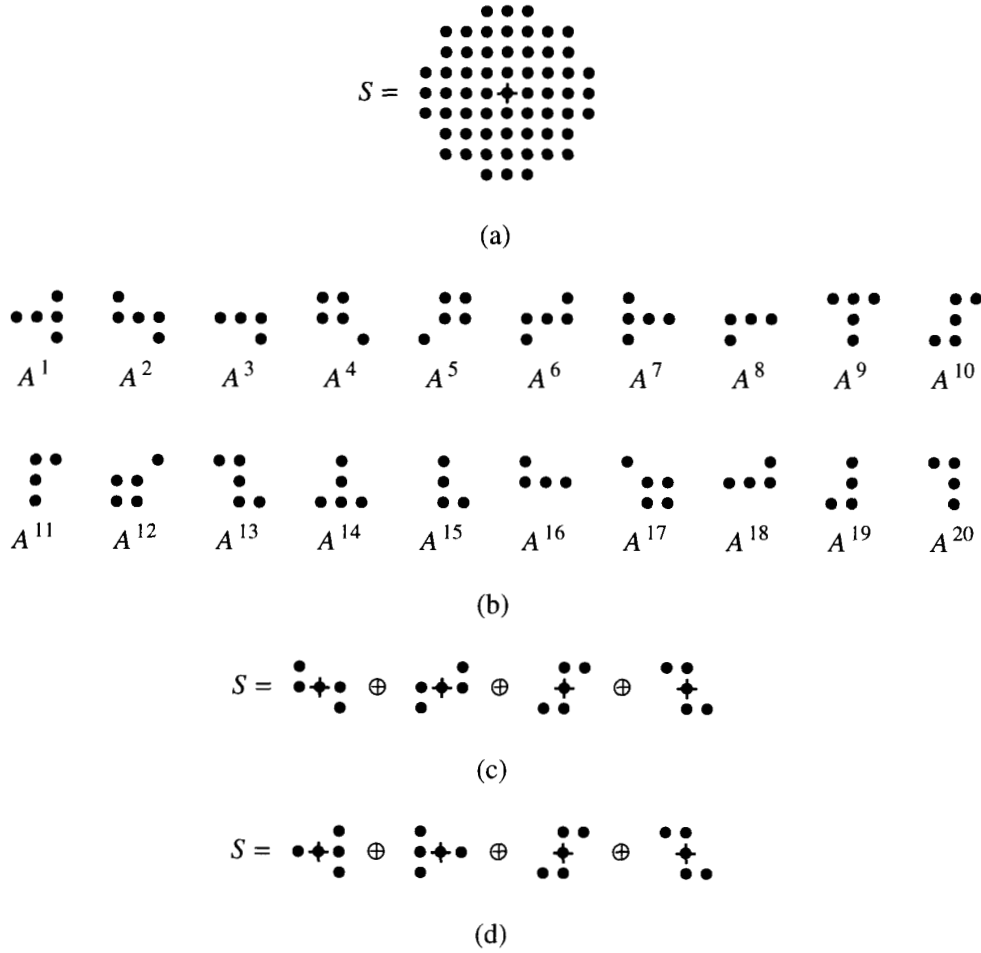


Fig. 7. Example 1: (a) A structuring element, S . (b) All 3×3 prime concave factors, A_i . (c) and (d) Two possible optimal decompositions of S . (+ indicates the origin.)

tion of S .

In general, the optimal decomposition into 3×3 elements is not unique, since the linear equations are underdetermined. To illustrate this, another \mathbf{X} satisfying Proposition 11 is sought for, resulting in

$$\mathbf{X} = [1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]^T.$$

In this case, the decomposition of S is

$$S = A^1 \oplus A^7 \oplus A^{10} \oplus A^{13}.$$

After optimization (Appendix B), the decomposition in Fig. 7(d) results. This is also an optimal decomposition because it requires four elements.

Example 2: Suppose that S is given by $S = Q_{L0}^3 12^4 Q_{r4}^3 Q_{r6} 6^2$ as in Fig. 8(a). It has been verified that $S \in \Phi$. Next, we find all the 3×3 prime concave factors for each concave boundary of S using Proposition 8 and Table I.

A^i is assigned to each of those factors as in Fig. 8(b), and V_i and d_i are constructed as follows:

$$V_1 = Q_{L0}, V_2 = Q_{r4}, V_3 = Q_{r6}, \\ d_1 = 0, d_2 = 1, d_3 = 2, d_4 = 4, d_5 = 6.$$

Then,

$$\Theta = \begin{bmatrix} 10000 \\ 01100 \\ 00011 \end{bmatrix}, \Omega = \begin{bmatrix} 01210 \\ 01001 \\ 20021 \\ 20000 \\ 02100 \end{bmatrix},$$

$$\mathbf{Y} = [1\ 1\ 1]^T \text{ and } \mathbf{Z} = [3\ 1\ 4\ 3\ 2]^T.$$

One possible \mathbf{X} satisfying Proposition 11 is

$$\mathbf{X} = [1\ 0\ 1\ 1\ 0]^T.$$

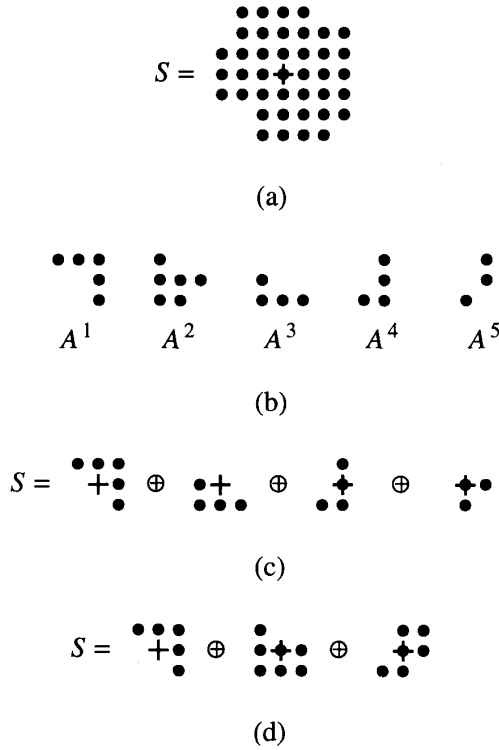


Fig. 8. Example 2: (a) A structuring element, S . (b) All 3×3 prime concave factors, A^i . (c) Sub-optimal decomposition of S . (d) An optimal decomposition of S . (+ indicates the origin.)

Therefore, the decomposition of S exists and is given by

$$S = A^1 \oplus A^3 \oplus A^4 \oplus B,$$

where $B = 146$. After optimization (Appendix B), we have the decomposition given in Fig. 8(c). Since this decomposition requires four 3×3 elements and the lower bound = $\max \{ l_{x_{\max}}(S), l_{x_{\min}}(S), l_{y_{\max}}(S), l_{y_{\min}}(S) \} = 3$, this is not guaranteed to be an optimal decomposition, therefore another \mathbf{X} is considered.

Another \mathbf{X} satisfying Proposition 11 is

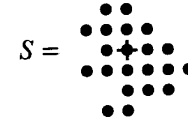
$$\mathbf{X} = [1 \ 0 \ 1 \ 0 \ 1]^T.$$

In this case, the decomposition of S is

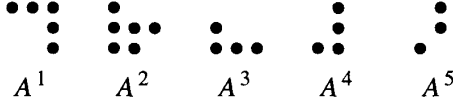
$$S = A^1 \oplus A^3 \oplus A^5 \oplus B,$$

where $B = 0246$, and the result of optimization is given in Fig. 8(d). Since it requires three 3×3 factors, it is an optimal decomposition of S .

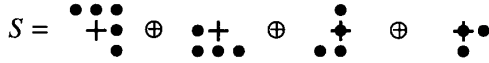
Example 3: Suppose that S is given by $S = Q_{10}0Q_{r1}13^345Q_{v7}$ as in Fig. 9(a). It has been verified that $S \in \Phi$, and all 3×3 prime concave factors of S are determined and A^i is assigned to each factor as in Fig. 9(b). V_i and d_i are constructed as follows:



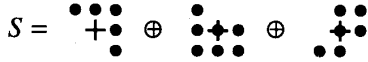
(a)



(b)



(c)



(d)

Fig. 9. Example 3: S is not decomposable. (a) A structuring element, S . (b) All 3×3 prime concave factors, A^i . (+ indicates the origin.)

$$V_1 = Q_{10}, V_2 = Q_{r1}, V_3 = Q_{v7}, \\ d_1 = 0, d_2 = 1, d_3 = 3, d_4 = 4, d_5 = 5.$$

Then,

$$\Theta = \begin{bmatrix} 100 \\ 010 \\ 011 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 000 \\ 101 \\ 111 \\ 110 \\ 000 \end{bmatrix},$$

$$\mathbf{Y} = [1 \ 1 \ 1]^T \text{ and } \mathbf{Z} = [1 \ 1 \ 3 \ 1 \ 1]^T.$$

A non-negative integer solution for $\Theta\mathbf{X} = \mathbf{Y}$ and $\Omega\mathbf{X} \leq \mathbf{Z}$ does not exist. Therefore, the decomposition of S does not exist, i.e., $S \notin D$.

Example 4: S is given by $S = Q_{L0}0^21^22^33^24Q_{L6}Q_{r7}$ as in Fig. 10(a). It has been verified that $S \in \Phi$, and all 3×3 prime concave factors of S are determined and A^i is assigned to each factor as in Fig. 10(b). V_i and d_i are constructed as follows:

$$V_1 = Q_{L0}, V_2 = Q_{L6}, V_3 = Q_{r7}, \\ d_1 = 0, d_2 = 1, d_3 = 2, d_4 = 3, d_5 = 4.$$

Then,

$$\Theta = \begin{bmatrix} 1000000 \\ 0110000 \\ 0111111 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0212211 \\ 0010011 \\ 1212110 \\ 1000101 \\ 1001010 \end{bmatrix},$$

$$\mathbf{Y} = [1 \ 1 \ 1]^T \text{ and } \mathbf{Z} = [2 \ 2 \ 2 \ 2 \ 1]^T.$$

One possible \mathbf{X} satisfying Proposition 11 is

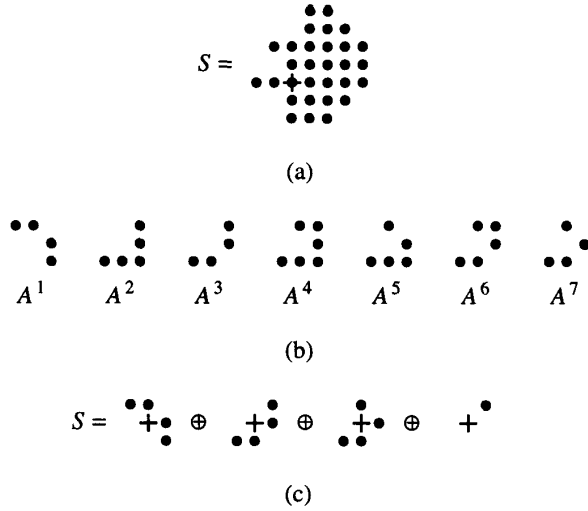


Fig. 10. Example 4: (a) A structuring element, S . (b) All 3×3 prime concave factors, A^i . (c) An optimal decomposition of S . (+ indicates the origin.)

$$\mathbf{X} = [1\ 0\ 1\ 0\ 0\ 0\ 1]^T.$$

Therefore, the decomposition of S exists. After optimization (Appendix B), the decomposition given in Fig. 10(c) results, which is an optimal decomposition of S since it requires four elements, the lower bound.

Example 5: S is given by $S = Q_{v1}Q_{r2}^23^24^25^26Q_{r7}7$ as in Fig. 11(a). It has been verified that $S \in \Phi$, and all 3×3 prime concave factors of S are determined and A^i is assigned to each factor as in Fig. 11(b). V_i and d_i are constructed as follows:

$$V_1 = Q_{v1}, V_2 = Q_{r2}, V_3 = Q_{r7}, \\ d_1 = 3, d_2 = 4, d_3 = 5, d_4 = 6, d_5 = 7.$$

Then,

$$\Theta = \begin{bmatrix} 110000 \\ 001111 \\ 000100 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 110000 \\ 012210 \\ 100011 \\ 011001 \\ 001010 \end{bmatrix},$$

$$\mathbf{Y} = [1\ 2\ 1]^T \text{ and } \mathbf{Z} = [2\ 2\ 2\ 1\ 1]^T.$$

One possible \mathbf{X} satisfying Proposition 11 is

$$\mathbf{X} = [1\ 0\ 0\ 1\ 0\ 1]^T,$$

and $B = 37$. After optimization (Appendix B), the decomposition given in Fig. 11(c) results. This \mathbf{X} is found to be the only solution for the linear equations (Equations (20) and (21)), so

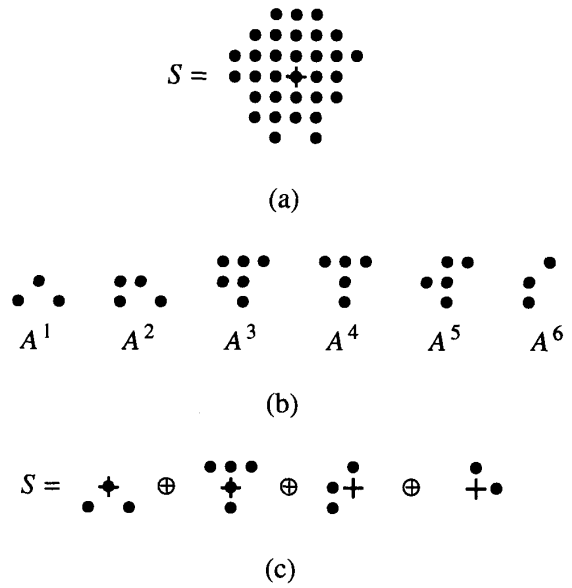


Fig. 11. Example 5: (a) A structuring element, S . (b) All 3×3 prime concave factors, A^i . (c) An optimal decomposition of S . (+ indicates the origin.)

it is an optimal decomposition, even though it requires more elements than the lower bound = $\max\{|x_{\max}(S)|, |x_{\min}(S)|, |y_{\max}(S)|, |y_{\min}(S)|\} = 3$.

VII. CONCLUSIONS

In this paper, we proposed an algorithm for optimal decomposition of arbitrarily shaped simply connected binary structuring elements into 3×3 elements. We first derived necessary conditions for decomposition and reduced the large image set to a smaller set by eliminating some images which are not decomposable. Next, we determined all the 3×3 prime concave factors for a given image using chain codes and formulated the decomposition problem into two constrained linear equations. The solution of the linear equations determines the decomposition of the image.

The main objectives of the decomposition of structuring elements are (i) to implement morphological operations on systems with small region of support by replacing the required large structuring elements with small elements, and (ii) to minimize the cost of the operation on systems with no practical limit on the size of region of support. The decomposition derived in this paper is optimized with respect to the number of 3×3 elements. Therefore, it provides an optimal implementation of morphological operations for systems which are based on a 3×3 region of support, such as the Cytocomputer. It also provides efficient implementation of morphological operations for systems where computational cost depends on the number of shifts, such as parallel array processors since the decomposition reduces the amount of shifts.

However, many structuring elements S are not decomposable into 3×3 elements, and additional processings must be first applied to S in order to partition S into a union of B^i given by

$$S = B^1 \cup B^2 \cup \dots \cup B^n$$

where

$$X \oplus S = \bigcup_{i=1}^n (X \oplus B^i).$$

If each B^i is decomposable, $X \oplus S$ can then be implemented after each B^i is decomposed into 3×3 elements. However, the partition is not unique and an optimal partition with respect to the number of 3×3 elements is yet to be investigated.

APPENDIX A: PROOFS OF PROPOSITIONS

In all figures in Appendix A, \bullet , \times and \cdot indicate a pixel (1), a non-pixel (0) and don't-care of a given image, respectively, and $+$ indicates the origin. The value (1 or 0) at a location marked by a small letter is specified in each image.

Proof of Proposition 1: To prove this proposition, we need to show that the boundary of S consists of Q_{T_i} and convex boundaries, and these boundary segments are arranged in a specific order defined in (2). Since B is 3×3 , the value at each location of S depends on a particular 3×3 local region of A , assuming that B contains the origin⁶. Therefore, the shape of each boundary segment of S depends on a particular local region of A and the shape of B . If any possible configuration of pixels in any local region of A , when dilated by any B , results in the boundary of S which satisfies (2), then no unsatisfactory boundary can stem from $A \oplus B$, which completes the proof. Instead of examining all possible local regions in A and all possible B , we only prove the following case — a local region containing Q_{U0} in A is dilated by B containing Q_{L0} . Other cases can be proved in a similar way.

Suppose A contains Q_{U0} . Then, since $A \in \Phi$, from (3), the neighborhood of Q_{U0} has the constraint shown in Fig. A1(a), where x , y , and z are don't-cares. Suppose B contains Q_{L0} as shown in Fig. A1(b). Since

$$S = \bigcup_{a \in A} (B)_a,$$

where $(B)_a = \{a + b \mid b \in B\}$, a particular location p is a pixel of S if $p \in (B)_a$ for some $a \in A$. In addition, p is a don't-care of S if $p \notin (B)_a$ for any $a \in A$, but $p \in (B)_{a'}$ for a don't-care a' of A , because the value of a' determines whether $p \in S$ or not. All other locations are non-pixels of S because they are not included in $(B)_a$ for any pixel or don't-care a of A . In this way, a portion of S , which stems from Q_{U0} in A and Q_{L0} in B , is constructed and shown in Fig. A1(c).

Now, we shall determine the shape of boundary of this region of S depending on the values of don't-cares in A . Consider

⁶ The origin can be located arbitrarily because the location of origin has no effect on the shape of the boundary.

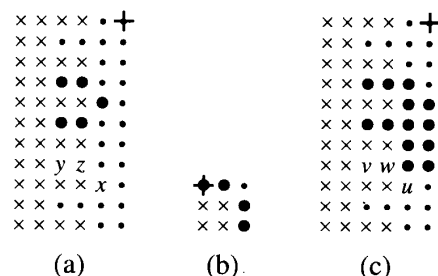


Fig. A1. Proof of Proposition 1. (a) A local region of A contains Q_{U0} . x , y , z are don't-cares. (b) B contains Q_{L0} . (c) A portion of $S = A \oplus B$ results from Q_{U0} in A and Q_{L0} in B . u , v , w are don't-cares.

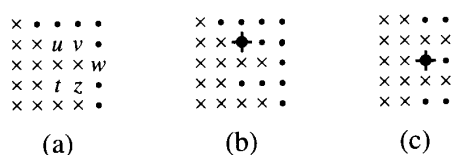


Fig. A2. Proof of Proposition 2. (a) S contains Q_{U0} . u , v , w , z , and t are pixels of S . (b) B has a pixel at $(0, 0)$. (c) A has a pixel at $(0, 0)$.

three don't-care locations x , y , and z in Fig. A1(a). When $y \in A$, then $z \in A$ since A is connected. In addition, since B contains the origin, $A \subset S$, so $v, w \in S$ in Fig. A1(c). In this case, the boundary of S becomes $Q_{U0}Q_{L0}$ regardless of the value of x . If $y \notin A$, then $v \notin S$ because there exists no pixel or don't-care a of A such that $v \in (B)_a$. Therefore, when $y \notin A$ and $z \in A$, the boundary becomes $Q_{U0}Q_{L0}$ because $v \notin S$ and $w \in S$. If $z \notin A$, then $x \notin A$ by the following reason: Since A is connected, x must be connected to Q_{U0} in A . However, since A cannot contain concave boundary other than Q_{T_i} , it is impossible to connect x to Q_{U0} in A if $z \notin A$. Therefore, $z \notin A$ implies $x \notin A$. Also, $x \notin A$ implies $u \notin S$ because there is no pixel or don't-care a of A such that $u \in (B)_a$ when $y, z \notin A$. In this case, the boundary becomes $Q_{U0}Q_{L0}$. In summary, for any possible combination of the values of x , y , and z in A , the boundary of S , which stems from Q_{U0} and its neighborhood in A and Q_{L0} in B , consists of Q_{T_i} . Moreover, Q_{T_i} in S is arranged in the same order as that in (2). Note that the rest of the boundary of S is not influenced by Q_{U0} in A because B is 3×3 . This completes the proof for the case when a local region containing Q_{U0} in A is dilated by B containing Q_{L0} . \square

Proof of Proposition 2: We shall prove Equations (7) and (8) when $i = 0$ only. All other cases can be proved in the same way.

(i) (Proof of (7)). Suppose S contains Q_{U0} . Since chain code of S has the form in (2) from Proposition 1, S has the constraint shown in Fig. A2(a) around Q_{U0} . To prove (7), we shall show that if $a_{U0} = 0$, then $b_{U0} \neq 0$. Suppose $a_{U0} = 0$. Since $u \in S$, there must exist $(x, y) \in B$ such that $u \in (A)_{(x, y)}$, and let $(x, y) = (0, 0)$ arbitrarily. Then, B has the constraint shown in Fig.

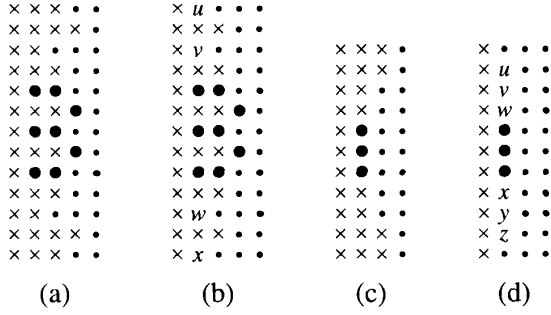


Fig. A3. Proof of Proposition 5. (a) A contains exactly two Q_{U0} 's. (b) S contains at least two Q_{U0} 's, $s_{U0} \geq 2$. u, v, w , and x are don't-cares. (c) A contains exactly two 6-directions, $a_6 = 2$. (d) S contains at least two 6-directions, $s_6 \geq 2$. u, v, w, x, y , and z are don't-cares.

A2(b) because $(A)_p$ for any location p marked by \times in Fig. A2(b) is not a subset of S , which is a contradiction to $S = A \oplus B$. Since we assume $(x, y) = (0, 0)$, $A \subset S$ and $u \in A$, but $t \notin A$ because A does not contain Q_{U0} . Thus, there must exist $(x', y') \in B$ such that $(x', y' \neq (0, 0))$ and $t \in (A)_{(x', y')}$. Now, we determine the possible locations of (x', y') . First, $y' \leq 0$; otherwise, $(A)_{(x', y')}$ must contain both t and another pixel whose y -coordinate is greater than that of u using only Q_{T7} , which is impossible under the constraint on S in Fig. A2(a). Second, since $t \in (A)_{(x', y')}$, if we shift both sides by (x', y') , we have $(t_x - x', t_y - y') \in (A)_{(0,0)} = A$. Finally, since B is 3×3 , $|x'| \leq 2$ and $|y'| \leq 2$. Then, we can show that only $(x', y') = (0, -2)$ satisfies all three conditions defined above. Therefore, $(x', y') = (0, -2) \in B$. Also, since B is connected, $(0, 0)$ and $(0, -2)$ must be connected, and under the constraint of B , this is possible only when B contains Q_{U0} . Therefore, $b_{U0} \neq 0$.

(ii) (Proof of (8)). Suppose $a_{U0} = 0$ and $b_{U0} \neq 0$. Since $(B)_a \subset S$ for all $a \in A$ and B contains Q_{U0} , there exists $(x, y) \in A$ such that Q_{U0} in $(B)_{(x, y)}$ coincides with Q_{U0} in S , and let $(x, y) = (0, 0)$ arbitrarily. Then, A has the constraint shown in Fig. A2(c); otherwise, $(B)_a$ for some $a \in A$ is not a subset of S . Since A is connected, $(0, 0)$ in Fig. A2(c) must be connected to other pixels in A . Because of the constraint on A in the chain code of A , the boundary connected to $(0, 0)$ from the left side must be Q_{J7}, Q_{L6}, Q_{R5} or direction 4, and the boundary connected to $(0, 0)$ from the right side must be Q_{T0} of any type or direction 0. If the boundary connected from the left is Q_{J7} , then $a_{V7} = a_{R7} = a_7 = 0$ because the boundary right after Q_{J7} is Q_{T0} or 0. In addition, since $a_{J7} \neq 0$, $a_6 = a_5 = a_{R6} = 0$ from (4). Therefore, $a_7 + a_6 + a_5 + a_{R7} + a_{R6} + a_{V7} = 0$. If the boundary from the left is Q_{L6}, Q_{R5} , or 4, we can derive the same result in the same way. Finally, since $B \in \Phi$ and $b_{U0} \neq 0$, $b_7 + b_6 + b_5 + b_{R7} + b_{R6} + b_{V7} = 0$ from (3). Therefore, (8) results.

The case when $a_{U0} \neq 0$ and $b_{U0} = 0$ can be proved in similar fashion, and when $a_{U0} \neq 0$ and $b_{U0} \neq 0$, (8) results directly from (3). \square

Proof of Proposition 3: We shall prove (3) and (5) when $i = 0$ only. Other case can be proved in a similar fashion.

(i) (Proof of (3)). Suppose $s_{U0} \neq 0$. From (8), $a_7 = a_6 = a_5 = a_{R7} = a_{R6} = a_{V7} = b_7 = b_6 = b_5 = b_{R7} = b_{R6} = b_{V7} = 0$. Then, $a_i + b_i = 0$ for $i = 5, 6, 7$, which violates (16) for $i = 5, 6, 7$. Therefore, S cannot contain convex boundary of directions 5, 6, and 7, so $s_i = 0$ for $i = 5, 6, 7$. Also, $a_i + a_{i-1} + a_{Ri} + b_i + b_{i-1} + b_{Ri} = 0$ for $i = 6, 7$, which violates (15) for $i = 6, 7$. Therefore, $s_{Ri} = 0$ for $i = 6, 7$. Finally, $a_5 + a_7 + a_{R6} + a_{R7} + a_{V7} + b_5 + b_7 + b_{R6} + b_{R7} + b_{V7} = 0$, which violates (13) for $i = 7$. Therefore, $s_{V7} = 0$. In summary, $s_5 = s_6 = s_7 = s_{R6} = s_{R7} = s_{V7} = 0$, so (3) results.

(ii) (Proof of (5)). Suppose $s_{L0} \neq 0$. Then, from (12), $a_7 = b_7 = 0$. Therefore, (16) for $i = 7$ is not satisfied, so $s_7 = 0$. Hence, (5) results. \square

Proof of Proposition 4: Since every 3×3 image is a member of Φ and Φ is closed under dilation by 3×3 image, every decomposable image is a member of Φ . \square

Proof of Proposition 5: We shall prove $s_{U0} = a_{U0} + b_{U0}$ and $s_6 = a_6 + b_6$ only. Other cases can be proved similarly.

(i) From Equations (2) and (3), the neighborhood of Q_{U0} in A has the constraint shown in Fig. A3(a), where $a_{U0} = 2$. Since Q_{U0} in A is not removed, $s_{U0} \geq a_{U0}$, and S has the constraint in Fig. A3(b). Consider four don't-cares u, v, w , and x in S . When B does not contain Q_{U0} , v and w cannot be pixels of S because $v, w \notin (B)_a$ for any pixel or don't-care a of A , therefore s_{U0} is equal to a_{U0} . When B contains Q_{U0} , v , or w , but not both, is a pixel of S . Then, $s_{U0} \geq a_{U0} + 1$ in order to connect v or w to pixels in S under the given constraint. However, u and x cannot be pixels of S , so $s_{U0} = a_{U0} + 1$. Therefore, in both cases, $s_{U0} = a_{U0} + b_{U0}$.

(ii) Fig. A3(c) shows the constraint on the neighborhood of direction 6 in A , where $a_6 = 2$. Fig. A3(d) shows the constraint on S because $s_6 \geq a_6$. Since $a_6 \neq 0$, Equations (8) for $i = 0$ and (14) for $i = 7$ are not satisfied; therefore, S cannot contain Q_{U0} and Q_{V7} . Consequently, from the assumption that no concave boundaries are removed, B cannot contain Q_{U0} and Q_{V7} , either. Consider six don't-cares in S . When $b_6 = 0$, w or x can be a pixel of S only when B contains Q_{V7} or Q_{U0} , which is impossible. Hence, $w, x \notin S$, and s_6 is equal to a_6 . When $b_6 = 1$, w or x , but not both, is a pixel of S , so $b_6 \geq a_6 + 1$. However, $v, y \notin S$, because B cannot contain Q_{V7} and Q_{U0} . In this case, $s_6 = a_6 + 1$. When $b_6 = 2$, $v, w \in S$ or $x, y \in S$, so $s_6 \geq a_6 + 2$. However, $u, z \notin S$ because B is 3×3 . In this case, $s_6 = a_6 + 2$. Therefore, in all three cases, $s_6 = a_6 + b_6$. \square

Proof of Proposition 6: We shall prove (17) when $i = 0$, that is, the case when Q_{U0} in A is removed after dilation by B . Other cases can be proved in a similar way. Suppose that $a_{U0} \neq 0$ but $s_{U0} = 0$. We assume that the origin of B is located at the starting pixel of the chain code, and the origin of A is located arbitrarily inside A . Suppose $b_7 + b_6 + b_5 + b_{R7} + b_{R6} + b_{V7} = 0$. Then, in chain code representation of B , two directions adjacent to the origin of B are 4 and 0. Therefore, when B is shifted to any pixel of A , no pixel of B can be placed at the indentation of Q_{U0} in A . Consequently, $S = A \oplus B$ contains Q_{U0}

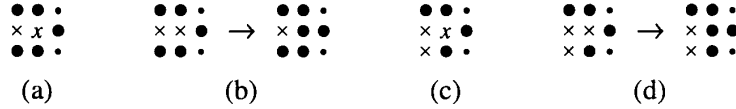


Fig. A4. Proof of Proposition 9. (a) B contains Q_{U0} . x is a non-pixel of B . For any $a \in A$, $(B)_a$ has a pixel at x because S does not contain Q_{U0} . (b) B is replaced by \hat{B} . (c) B contains Q_{U0} . x is a non-pixel of B . For any $a \in A$, $(B)_a$ has a pixel at x because S does not contain Q_{U0} and Q_{U0} . (d) B is replaced by \hat{B} .

which is a contradiction to the assumption of $s_{U0} = 0$. Therefore, (17) results. \square

Proof of Proposition 7 : Let $P = A \oplus B \in \Phi$. Suppose a particular Q_{Ti} in A is removed due to dilation by B ; then, (17), (18), or (19) for B , depending on the type of concave boundary Q_{Ti} in A , must be satisfied. Hence, (17) (18), or (19) for S must be satisfied because $s_i = a_i + b_i$ and $s_{Ti} = a_{Ti} + b_{Ti}$. In addition, $s_{Ti} \neq 0$ because $a_{Ti} \neq 0$ and $s_{Ti} = a_{Ti} + b_{Ti}$. These two conditions on s_i and s_{Ti} , however, mean that $S \notin \Phi$ from (3), (4), (5), or (6), which is a contradiction to the assumption $S \in \Phi$. Therefore, all Q_{Ti} in A must be contained in P . Similarly, all Q_{Ti} in B are contained in P . Then, from Proposition 5, $p_i = a_i + b_i$ and $p_{Ti} = a_{Ti} + b_{Ti}$. Therefore, $s_{Ti} = p_{Ti}$ and $s_i = p_i$, which means that the chain code of $P = A \oplus B$ is identical to that of S and the boundary of $A \oplus B$ is identical to that of S . \square

Proof of Proposition 8: From Proposition 7, the boundary of S is identical to that of $A \oplus B$. Since $A \oplus B$ is simply connected, $S = A \oplus B$, so B is a factor of S . \square

Proof of Proposition 9: We shall prove the cases for Q_{U0} and Q_{U0} ; other cases can be proved similarly. First, we shall prove that B in $S = A \oplus B$ can be replaced by \hat{B} such that $S = A \oplus \hat{B}$. (i) Suppose B contains Q_{U0} and S does not, that is, $b_{U0} \neq 0$ and $s_{U0} = 0$. Since

$$S = A \oplus B = \bigcup_{a \in A} (B)_a,$$

$(B)_a \subset S$ for all $a \in A$. Since S does not contain Q_{U0} , a pixel of S must be located at x in each $(B)_a$ as defined in Fig. A4(a). Therefore, $(B)_a \subset S$ implies $(\hat{B})_a \subset S$ for all $a \in A$, where \hat{B} corresponds to B after Q_{U0} is replaced by Q_{U0} (see Fig. A4(b) for example). Moreover, since $B \subset \hat{B}$,

$$S = \bigcup_{a \in A} (B)_a \subset \bigcup_{a \in A} (\hat{B})_a \subset S.$$

Therefore,

$$S = \bigcup_{a \in A} (\hat{B})_a \oplus A \oplus \hat{B}.$$

(ii) Suppose B contains Q_{U0} and S does not. Then, from (18), $a_7 + a_6 + a_{r7} \neq 0$, which violates (8). Therefore, S does not contain Q_{U0} . Consequently, a pixel of S must be located at x in each $(B)_a$ as defined in Fig. A4(c); otherwise, S contains Q_{U0} . Therefore, $(B)_a \subset S$ implies $(\hat{B})_a \subset S$ for all $a \in A$, where \hat{B} corresponds to B after Q_{U0} is replaced by Q_{U0} (see Fig. A4(d) for example). Therefore,

$$S = \bigcup_{a \in A} (\hat{B})_a \oplus A \oplus \hat{B}.$$

Now, we shall prove that all concave boundaries in \hat{B} are contained in S . Suppose that a concave boundary in \hat{B} is not contained in S . Then replacement of \hat{B} can be repeated until all concave boundaries in the new \hat{B} are contained in S or the new \hat{B} has no concave boundary. Therefore, all concave boundaries in \hat{B} are contained in S . \square

Proof of Proposition 10: Since S is decomposable from Proposition 9, S can be represented by $S = A^1 \oplus \dots \oplus A^n$, where A^k is 3×3 or less and all concave boundaries in A^k are contained in S . Then, from Proposition 5,

$$s_{Ti} = \sum_{k=1}^n a_{Ti}^k.$$

Therefore, $s_{Ti} \neq 0$ implies that there exists k such that $a_{Ti}^k \neq 0$, so at least one factor contains Q_{Ti} . \square

Proof of Proposition 11: (i) (Sufficient). Let $P = x_1 A^1 \oplus x_2 A^2 \oplus \dots \oplus x_n A^n$. The product of i^{th} row of Θ and \mathbf{X} in (20) is the number of concave boundary V_i 's in P from Proposition 5 because all concave boundaries in A^i are contained in P . Therefore, (20) implies that $p_{Ti} = s_{Ti}$. The product of i^{th} row of Ω and \mathbf{X} in (21) is the number of direction d_i 's in P from Proposition 5. Therefore, (21) implies that $p_i \leq s_i$. Let $b_i = s_i - p_i$. Then B created by b_i defines a convex image in Φ , and $s_i = p_i + b_i$ and $s_{Ti} = p_{Ti} + b_{Ti}$, where $b_{Ti} = 0$. Therefore, from Proposition 7, the boundaries of S and $P \oplus B$ are identical. In addition, $P \oplus B$ is simply connected, so is equal to S . Also, B is convex and always decomposable into 3×3 elements [5]. Therefore, S is decomposable.

(ii) (Necessary). Suppose S is decomposable, $S = x_1 A^1 \oplus \dots \oplus x_n A^n \oplus B$, where A^i is a 3×3 prime concave factor whose concave boundaries are contained in S , and B is a convex factor of arbitrary size. Then,

$$s_{Ti} = \sum_{k=1}^n x_k a_{Ti}^k \text{ and } s_i = \left[\sum_{k=1}^n x_k a_i^k \right] + b_i$$

from Proposition 5. Therefore, there exists an \mathbf{X} such that Equations (20) and (21) are satisfied and $x_1 A^1 \oplus \dots \oplus x_n A^n \oplus B$ is simply connected. \square

APPENDIX B: OPTIMIZATION PROCEDURE

This appendix gives a procedure to find the decomposition of convex factor B , to locate the origin of each factor, and to minimize the number of 3×3 elements.

- 1) Replace Step 2 of Algorithm B in [5] by the following: Determine the decomposition of B using Algorithm A in [5]. For all concave factors $\{A^i\}$ and convex factors derived from B , put 1×2 factors in the class C_1 , 2×1 factors in C_2 , 2×2 factors in C_3 , 2×3 factors in C_4 , 3×2 factors in C_5 , and 3×3 factors in C_6 .
- 2) Continue Algorithm B of [5].

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